International Journal of Theoretical Physics, Vol. 43, No. 5, May 2004 (© 2004)

Constructions of Some Quantum Structures and Fuzzy Effect Space

Shang Yun,[1,3] Li Yongming,[1] and Chen Maoyin[2]

Received April 23, 2004

Quantum structures like effect algebras, σ-effect algebras, orthoalgebras, orthomodular posets, and σ-orthomodular posets are constructed by use of special fuzzy sets on posets. The concept of fuzzy effect space is introduced and a representation of a lattice effect algebra with a strong order determining system of states by means of fuzzy effect space is established.

KEY WORDS: effect algebras; orthoalgebras; orthomodular posets; order determining system of states; fuzzy effect space.

1. INTRODUCTION

Since Birkhoff and Von Neumann in 1936 proposed the problem of the logic of quantum mechanics, a number of different mathematical models have been constructed and studied to reflect various aspects of quantum mechanics, among which the lattice of all closed subspaces of a separable infinite dimensional Hilbert space (namely, an orthomodular lattice) plays an important role as a main model (Dvurečenskij and Pulmannová, 2000; Miklós, 1998). In the past decades, with the development of the theory of quantum logics, new algebraic structures have been proposed as their models. Foulis and Bennet (1994) defined effect algebras while Kôpka and Chovanec (1994) introduced D-posets equivalent to effect algebras. In addition, Giuntini and Greuling (1989) introduced orthoalgebras. These works can be regarded as generalization of quantum logics, i.e., orthomodular lattice or orthomodular poset (Kalmbach, 1983). In the study of quantum logics, the state is considered as one of basic objects. Since the state is [0, 1]-valued function, it

[1] College of Mathematics and Information Science, Shaanxi Normal University, Xi'an Peoples's Republic of China.
[2] Department of Automation, Tsinghua University, Beijing , Peoples's Republic of China.
[3] To whom correspondence should be addressed at College of Mathematics, and Information Science, Shaanxi Normal University, Xi'an 710062, People's Republic of China, e-mail: shangyun602@163.com.

0020-7748/04/0500-1219/0

combines both algebraic and fuzzy ideas, which further leads to the generation of fuzzy quantum logics (Kôpka, 1992; Mesiar, 1994; Pykacz, 1992, 1994). From the angle of fuzzy sets, Dvurečenskij (1999) gave fuzzy representations of some quantum structures by the states on them in detail. In this paper we give constructions of some quantum structures via fuzzy sets on posets from the universe point of view. These include constructions of effect algebras, σ-effect algebras, orthoalgebras, orthomodular posets, and σ-orthomodular posets by use of special fuzzy sets on the posets. In the end, we introduce the notions of fuzzy effect space, and establish a representation of a lattice effect algebra with a strong order determining system of states by means of fuzzy effect space.

2. CONSTRUCTION OF EFFECT ALGEBRAS

Definition 2.1. (Foulis and Bennett, 1994). An effect algebra is a set P with two particular elements 0, 1 ($0 \neq 1$), and with a partial binary operation $\oplus : P \times P \longrightarrow P$ such that for all $a, b, c \in P$ we have

(EAi) If $a \oplus b \in P$, then $b \oplus a \in P$ and $a \oplus b = b \oplus a$ (commutativity);
(EAii) If $b \oplus c \in P$ and $a \oplus (b \oplus c) \in P$, then $a \oplus b \in P$ and $(a \oplus b) \oplus c \in P$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);
(EAiii) For any $a \in P$ there is a unique $b \in P$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);
(EAiv) If $1 \oplus a$ is defined, then $a = 0$ (zero-one law).
If the assumption of (EAii) is satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in P.

Remark 2.2. Let a and b be two elements of an effect algebra P.

(i) $a \leq b$ iff there exists an element $c \in P$ such that $a \oplus c = b$. If (P; $\leq$) is a lattice, then P is called a lattice effect algebra.
(ii) $a \perp b$ iff $a \leq b^{\perp}$ iff $a \oplus b$ is defined in P.
(iii) b is the orthocomplement of a iff b is a unique element of P such that $a \oplus b = 1$ and it is written as $a^{\perp}$.

We say that a finite sequence $F = \{a_1, a_2, \ldots, a_n\}$ in P is $\oplus$-orthogonal if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists in P. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $a_1 \oplus a_2 \oplus \cdots a_{n-1}$ exists and $(a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \perp a_n$. An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of P is $\oplus$-orthogonal iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$ is $\oplus$-orthogonal. An $\oplus$-orthogonal system $G = \{a_i : i \in I\}$ of P has a $\oplus$-sum in P, denoted by $\oplus_{i \in I} a_i$ iff in P there exists the join $\oplus_{i \in I} a_i = \vee_J \oplus_{i \in J} a_i$, where J runs over all finite subsets in I. An effect algebra P is a σ-effect algebra (complete effect algebra) if $\oplus_{i\in I} a_i$ belongs to P for any countable (arbitrary) system $\{a_i : i \in I\}$ of $\oplus$-orthogonal elements from P.

Example 2.3. (Dvurečenskij, 1999). Let the closed interval [0, 1] be ordered by the natural way. For two numbers $a, b \in [0, 1]$, we define $a \oplus b$ iff $a + b \leq 1$ and put then $a \oplus b = a + b$. Then [0, 1] is a lattice effect algebra. In addition, we recall that $\{a_s\}$ is $\oplus$-orthogonal iff $\{a_s\}$ is summable and $\Sigma_s a_s \leq 1$. Hence, any $\oplus$-orthogonal system has the sum in it and $\oplus_s a_s = \Sigma_s a_s$. Obviously, [0, 1] is σ-effect algebra.

Definition 2.4. (Dvurečenskij, 1999). A real-valued mapping m on an effect algebra P is said to be a state if (i) $m(1) = 1$, and (ii) $m(a \oplus b) = m(a) + m(b)$, a, $b \in P$. A state m is said to be a σ-additive state if $m(\oplus_{i\in I} a_i) = \Sigma_{i\in I} m(a_i)$ holds for any countable index set I, whenever $\oplus_{i\in I} a_i$ exists in P.

Definition 2.5. (Pykacz, 1992). A nonvoid system of states φ on P is said to be order determining if, for $a, b \in P$, $a \leq b$ iff $\sigma(a) \leq \sigma(b)$ for any $\sigma \in \varphi$.

Theorem 2.6. *Let P be a poset with the least element* 0 *and the largest element*

1. *A set φ of maps from P into* [0, 1] *satisfy the following conditions:*
   (i) For $a, b \in P$, $a \leq b$ iff $\sigma(a) \leq \sigma(b)$, $\sigma(0) = 0$, $\sigma(1) = 1$, for any $\sigma \in \varphi$.
   (ii) If for $d_1, d_2 \in P$, $\sigma(d_1) + \sigma(d_2) \leq 1$ for any $\sigma \in \varphi$, then there exists $b \in P$ such that $\sigma(b) + \sigma(d_1) + \sigma(d_2) = 1$, for all $\sigma \in \varphi$.

Then $P = (P, \oplus, 0, 1)$, where $a \oplus b$ is defined iff $\sigma(a) + \sigma(b) \leq 1$ $(a, b \in P, \sigma \in \varphi)$ and we put $\sigma(a \oplus b) = \sigma(a) + \sigma(b)$, is an effect algebra. Moreover, φ is an order determining system of states on $(P, \oplus, 0, 1)$.

Conversely, if P is an effect algebra with an order determining system of states φ, then φ satisfy the above conditions.

Proof: Suppose $a \in P$, $\sigma(a) \leq 1$, for any $\sigma \in \varphi$, by (i) and (ii), then there exists a unique $a^{\perp} \in P$, such that $\sigma(a^{\perp}) = 1 - \sigma(a)$. Now, let $a, b \in P$. If $\sigma(a) + \sigma(b) \leq 1$ for any $\sigma \in \varphi$, by (i), (ii), then there exists a unique $c \in P$ such that $\sigma(a) + \sigma(b) + \sigma(c) = 1$, hence, $\sigma(a) + \sigma(b) = \sigma(c^{\perp})$, for any $\sigma \in \varphi$. Let $a \oplus b = c^{\perp}$, then $a \oplus b$ is well defined.

(EAi) If $a \oplus b \in P$, then for any $\sigma \in \varphi$, $\sigma(a) + \sigma(b) \leq 1$ such that $\sigma(a \oplus b) = \sigma(a) + \sigma(b) = \sigma(b \oplus a)$. So by (i) $a \oplus b = b \oplus a$.

(EAii) If $b \oplus c \in P$ and $a \oplus (b \oplus c) \in P$, then $\sigma(b) + \sigma(c) \leq 1$, $\sigma(a) + \sigma(b \oplus c) \leq 1$. Since $\sigma(a) + \sigma(b) + \sigma(c) \leq 1$ which implies that

$\sigma(a) + \sigma(b) \leq 1$. So $a \oplus b$ is defined and $\sigma(a \oplus b) + \sigma(c) \leq 1$ such that $(a \oplus b) \oplus c$ is defined.

Obviously, $\sigma((a \oplus b) \oplus c) = \sigma(a \oplus (b \oplus c))$. So $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(EAiii) For any $a \in P, \sigma \in \varphi, \sigma(a) + \sigma(0) \leq 1$ by (ii) there exists $b \in P$ such that $\sigma(a) + \sigma(b) = 1$. Obviously b is unique. So $a \oplus b$ is defined and $\sigma(a \oplus b) = \sigma(1)$, then $a \oplus b = 1$ (orthocomplementation).

(EAiv) If $1 \oplus a$ is defined, then for any $\sigma \in \varphi$, $\sigma(1 \oplus a) = \sigma(1) + \sigma(a) \leq 1$. So $\sigma(a) = 0$ then $a = 0$.

So $P = (P, \oplus, 0, 1)$ is an effect algebra. Obviously, φ is an order determining system of states.

Conversely, let $P = (P, \oplus, 0, 1)$ be an effect algebra with an order determining system of states φ. Clearly, (i) is satisfied. If for $a, b \in P, \sigma(a) + \sigma(b) \leq 1, \sigma \in \varphi$, then $\sigma(a) \leq \sigma(b^{\perp})$ such that $a \leq b^{\perp}$. Hence, there exists $c \in P$ such that $a \oplus c = b^{\perp}$ and $a \perp c$. Then $\sigma(a \oplus c) = \sigma(b^{\perp})$ i.e., $\sigma(a) + \sigma(c) = 1 - \sigma(b)$ such that $\sigma(a) + \sigma(b) + \sigma(c) = 1$. □

Theorem 2.7. *Let P be a poset with the least element 0 and the largest element*

1. *A set φ of maps from P into* [0,1] *satisfy the following conditions:*
   (i) *For $a, b \in P$, $a \leq b$ iff $\sigma(a) \leq \sigma(b)$ and $\sigma(0) = 0, \sigma(1) = 1, \sigma \in \varphi$;*
   (ii) *If for $d_1, d_2 \in P$, $\sigma(d_1) + \sigma(d_2) \leq 1$ for any $\sigma \in \varphi$, then there exists $b \in P$ such that $\sigma(b) + \sigma(d_1) + \sigma(d_2) = 1, \sigma \in \varphi$;*
   (iii) *If $\sigma(a_1) \leq \sigma(a_2) \leq \cdots \leq \sigma(a_i) \leq \cdots$, then there exists $b \in P$ such that $\sigma(b) = \vee_{i \in I}\sigma(a_i)$.*

Then $(P, \oplus, 0, 1)$ is a σ-effect algebra, where $a \oplus b$ is defined iff $\sigma(a) + \sigma(b) \leq 1(a, b \in P)$ for any $\sigma \in \varphi$, and we put $\sigma(a \oplus b) = \sigma(a) + \sigma(b)$. φ is the σ-additive order determining system of states.

Conversely, if $(P, \oplus, 0, 1)$ is a σ-effect algebra with an σ-additive order determining system states. Then the states of P satisfy the above conditions.

Proof: By theorem 2.6 we note that $(P, \oplus, 0, 1)$ is an effect algebra with an order determining system of states. We only prove $(P, \oplus, 0, 1)$ is σ-complete. For any countable system $a_i, i = 1, 2, \ldots$ of orthogonal elements from P, and for any finite subset J of N, the set of natural numbers, there exists a natural number n such that for any element $j \in J$, j is smaller than or equal to n. Then it is obvious that $\oplus_{j \in J} a_j \leq \oplus_{i=1}^{n} a_i$. So in order to prove that P is σ-complete, we only prove that $\vee_{n \in N}(\oplus_{i=1}^{n} a_i)$ is in P. Since $a_1 \leq a_1 \oplus a_2 \leq a_1 \oplus a_2 \oplus a_3 \leq \cdots$, then $\sigma(a_1) \leq \sigma(a_1 \oplus a_2) \leq \sigma(a_1 \oplus a_2 \oplus a_3) \leq \cdots$, by (iii) there exists $b \in P$ such that $\sigma(b) = \vee_{n \in N}\sigma(\oplus_{i=1}^{n} a_i)$. We conclude that $b = \vee_{n \in N}(\oplus_{i=1}^{n} a_i)$. Indeed, it is easy to say that $\oplus_{i=1}^{n} a_i \leq b$, for all $n \in N$. For all c, all $n \in N$, $\oplus_{i=1}^{n} a_i \leq c$, then $\sigma(\oplus_{i=1}^{n} a_i) \le \sigma(c)$, hence, $\vee_{n\in N}\sigma(\oplus_{i=1}^{n} a_i) \le \sigma(c)$, i.e., $\sigma(b) \le \sigma(c)$, which implies that $b \le c$. This shows that P is σ-complete. Obviously, φ is order determining system of σ-additive states on P. Indeed, from the above proof, for any countable system $\{a_i : i \in I\}$ of $\oplus$-orthogonal elements in P, $\sigma(\oplus_{i\in I} a_i) = \vee_{n\in N}\sigma(\oplus_{i=1}^{n} a_i) = \vee_{n\in N}\Sigma_{i=1}^{n}\sigma(a_i) = \Sigma_{i=1}^{\infty}\sigma(a_i)$, since [0, 1] is σ-effect algebra.

Conversely, by Theorem 2.6, (i), (ii) is satisfied. Let $\sigma(a_1) \le \sigma(a_2) \le \cdots \le \sigma(a_i) \le \cdots$, which implies that $a_1 \le a_2 \le a_3 \le \cdots$, then there exists $b_1, b_2, b_3, \cdots$ such that $a_1 = b_1, b_1 \oplus b_2 = a_2, a_2 \oplus b_3 = a_3, \cdots$. Thus, we obtain a countable orthogonal sequence $b_1, b_2, \cdots$ of P. Since P is σ-complete, it is easy to see that $\oplus_{i=1}^{\infty} b_i = \vee_{n\in N} \oplus_{i=1}^{n} b_i = V_{n\in N}\, a_n$. Hence, $\sigma(\oplus_{i=1}^{\infty} b_i) = \Sigma_{i=1}^{\infty}\sigma(b_i) = \vee_{n\in N}\Sigma_{i=1}^{n}\sigma(b_i) = \vee_{n\in N}\sigma(a_n)$, i.e., (iii)is true. □

3. CONSTRUCTION OF ORTHOALGEBRAS

Definition 3.6. (Giuntini and Greuling, 1989). An orthoalgebra is a set P with two particular elements 0, $1 (0 \ne 1)$, and with a partial binary operation $\oplus : P \times P \longrightarrow P$ such that for all $a, b, c \in P$ we have

(OAi) If $a \oplus b \in P$, then $b \oplus a \in P$ and $a \oplus b = b \oplus a$ (commutativity);
(OAii) If $b \oplus c \in P$ and $a \oplus (b \oplus c) \in P$, then $a \oplus b \in P$ and $(a \oplus b) \oplus c \in P$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);
(OAiii) For any $a \in P$ there is a unique $b \in P$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);
(OAiv) If $a \oplus a$ is defined, then $a = 0$ (consistency).

Similarly as for effect algebras we introduce a partial order $\le$ on P, and we write $a \perp b$ iff $a \oplus b$ exists in P. A state and an order determining system of states on orthoalgebras are defined as those for effect algebras.

Theorem 3.8. *Let P be a poset with the least element* 0 *and the largest element*

1. A set φ of maps from P into [0, 1] *satisfy the following conditions*:
  (i) For $a, b \in P, a \le b$ iff $\sigma(a) \le \sigma(b), \sigma(0) = 0, \sigma(1) = 1$ for any $\sigma \in \varphi$.
  (ii) If for $d_1, d_2 \in P, \sigma(d_1) + \sigma(d_2) \le 1$ for any $\sigma \in \varphi$, then there exists $b \in P$ such that $\sigma(b) + \sigma(d_1) + \sigma(d_2) = 1, \sigma \in \varphi$.
  (iii) If $a \in P, \sigma(a) + \sigma(a) \le 1$ for any $\sigma \in \varphi$, then $a = 0$.

Then $P = (P, \oplus, 0, 1)$, where $a \oplus b$ is defined iff $\sigma(a) + \sigma(b) \le 1 (a, b \in P)$ for any $\sigma \in \varphi$ and we put $\sigma(a \oplus b) = \sigma(a) + \sigma(b)$, is an orthoalgebra. And φ is order determining system of states on P.

Conversely, if P is an orthoalgebra with an order determining system of states φ, then φ satisfies the above conditions.

The proof is similar to the proof of Theorem 2.6. Condition (iii) implies that P is an orthoalgebra.

4. CONSTRUCTION OF ORTHOMODULAR POSETS

Definition 4.7. (Kalmbach, 1983). An orthomodular poset is a poset $P = (P, \leq, \perp, 0, 1)$ $(0 \neq 1)$ satisfying the following conditions:

(OMi) $(a^{\perp})^{\perp} = a$ for any $a \in P$.
(OMii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$, $a, b \in P$.
(OMiii) $a \vee a^{\perp} = 1$ for any $a \in P$, where $\vee$ denotes the least upper bound.
(OMiv) $a \vee b \in P$, whenever $a, b \in P$ and $a \perp b$.
(OMv) if $a \leq b$, $a, b \in P$, then $b = a \vee (b \wedge a^{\perp})$.

If we change the condition (OMiv) to $\vee_{i=1}^{\infty} a_i \in P$, whenever $a_i \perp a_j$, $i \neq j$, $a_i \in P$, $i \geq 1$. We call P a σ-orthomodular poset. A state on an orthomodular poset P is a mapping $\sigma: P \longrightarrow [0, 1]$ such that $\sigma(1) = 1$ and $\sigma(a \vee b) = \sigma(a) + \sigma(b)$ whenever $a \leq b^{\perp}$. Similarly, we define a σ-additive state on a σ-orthomodular poset P, i.e., $\sigma: P \longrightarrow [0, 1]$ such that (i) $\sigma(1) = 1$, and (ii) $\sigma(\vee_{i=1}^{\infty} a_i) = \Sigma_{i=1}^{\infty} \sigma(a_i)$ whenever $a_i \leq a_j^{\perp}$ for $i \neq j$.

Theorem 4.9. *Let P be a poset with the least element* 0 *and the largest element*

1. *A set φ of maps from P into* $[0, 1]$ *satisfy the following conditions:*
   (i) *For $a, b \in P$, $a \leq b$ iff $\sigma(a) \leq \sigma(b)$, and $\sigma(0) = 0, \sigma(1) = 1$ for any $\sigma \in \varphi$.*
   (ii) *For any $d_1, d_2, d_3 \in P$ if $\sigma(d_i) + \sigma(d_j) \leq 1, i \neq j$, for any $\sigma \in \varphi$, then there exists $b \in P$ such that $\sigma(b) + \sigma(d_1) + \sigma(d_2) + \sigma(d_3) = 1$, for any $\sigma \in \varphi$.*

Then $P = (P, \leq, \perp)$, whenever for $a \in P$, $a^{\perp}$ is the unique element of P satisfying $\sigma(a) + \sigma(a^{\perp}) = 1$, for any $\sigma \in \varphi$, is an orthomodular poset and φ is an order determining system of states on P.

Conversely, if P is an orthomodular poset with an order determining system of states φ, then φ satisfies the above conditions.

Proof: Suppose $a \in P$. If $\sigma(a) \leq 1$, for any $\sigma \in \varphi$, let $b, c = 0$, then $\sigma(a) + \sigma(b) + \sigma(c) \leq 1$, for any $\sigma \in \varphi$. By (ii), then there exists a $d \in P$, such that $\sigma(a) + \sigma(d) = 1$. Obviously, d is unique. Hence, let $a^{\perp} = d$, then $a^{\perp}$ is well defined.

(OMi) For $a^\perp$, $\sigma((a^\perp)^\perp) + \sigma(a^\perp) = 1$. So $\sigma(a) + \sigma(a^\perp) = \sigma(a^{\perp\perp}) + \sigma(a^\perp)$. Then $\sigma(a) = \sigma(a^{\perp\perp})$, by (i), $a = a^{\perp\perp}$.

(OMii) If $a \le b$, then $b^\perp \le a^\perp$. Let $a \le b$, by (i), $\sigma(a) \le \sigma(b)$ for any $\sigma \in \varphi$, since $\sigma(b) + \sigma(b^\perp) = 1$, then $\sigma(a) + 1 - \sigma(b) \le 1$, i.e., $1 - \sigma(b) \le 1 - \sigma(a)$ which implies that $\sigma(b^\perp) \le \sigma(a^\perp)$, then $b^\perp \le a^\perp$.

(OMiii) $a \vee a^\perp = 1$ for any $a \in P$, where $\vee$ denotes the least upper bound. Indeed, if there is an element $b \in P$ such that $a, a^\perp \le b$, then $\sigma(a) + \sigma(a^\perp) \le 1, \sigma(a^\perp) + \sigma(b^\perp) \le 1,\ \sigma(a) + \sigma(b^\perp) \le 1$, by (ii), there exists $d \in P$ such that $\sigma(d) + \sigma(a^\perp) + \sigma(b^\perp) + \sigma(a) = 1$, so $\sigma(d) = \sigma(b^\perp) = 0$ which implies that $b = 1$. Then $P = (P, \le, \perp)$ is an orthoposet.

(OMiv) $a \vee b \in P$, whenever $a, b \in P$ and $a \perp b$. Since P is an orthoposet, $a, b \in P$, then $a \perp b$ iff $a \le b^\perp$. Let $a \perp b$, by (i), $\sigma(a) + \sigma(b) \le 1$ for any $\sigma \in \varphi$. Obviously $\sigma(a) + \sigma(0) \le 1, \sigma(b) + \sigma(0) \le 1$, by (ii), there exists $c \in P$ such that $\sigma(a) + \sigma(b) + \sigma(c^\perp) + \sigma(0) = 1$, i.e., $\sigma(a) + \sigma(b) + \sigma(c^\perp) = 1$. We assert $c = a \vee b$. If for $d \in P, a \le d, b \le d$, then $\sigma(a) + \sigma(d^\perp) \le 1, \sigma(b) + \sigma(d^\perp) \le 1, \sigma(a) + \sigma(b) \le 1$, by (ii), there exists $e \in P$ such that $\sigma(e) + \sigma(a) + \sigma(b) + \sigma(d^\perp) = 1$, hence $\sigma(a) + \sigma(b) = \sigma(c) = \sigma(d) - \sigma(e) \le \sigma(d)$. So $c \le d$, then $c = a \vee b$.

(OMv) If $a \le b, a, b \in P$, then $a \perp b^\perp$, by (OMiv), $a \vee b^\perp \in P$, then $b^\perp \perp (a \vee b^\perp)^\perp$ and $a \perp (a \vee b^\perp)^\perp$. By (ii), there exists $e \in P$ such that $\sigma(e) + \sigma(b^\perp) + \sigma(a) + \sigma((a \vee b^\perp)^\perp) = 1$ for any $\sigma \in \varphi$, hence, $\sigma(e) + \sigma(b^\perp) + \sigma(a) = \sigma(a \vee b^\perp)$. By the proof of (OMiv), $\sigma(e) + \sigma(b^\perp) + \sigma(a) \simeq \sigma(a) + \sigma(b^\perp)$. So $\sigma(e) = 0$, then $e = 0$ such that $\sigma(b) = \sigma(a \vee (b^\perp \vee a)^\perp)$. Hence $b = a \vee (b^\perp \vee a)^\perp$, then $P = (P, \le, \perp)$ is an orthomodular poset.

Obviously, φ is an order determining system of states on P.

Conversely, $P = (P, \le, \perp)$ is an orthomodular poset. φ is the order determining system of states on P. Obviously, (i) is satisfied. If for $d_1, d_2, d_3 \in P, \sigma(d_i) + \sigma(d_j) \le 1, i \ne j$, for any $\sigma \in \varphi$. Then $\sigma(d_i) \le 1 - \sigma(d_j) = \sigma(d_j^\perp)$, then $d_i \le d_j^\perp$. So $d_i \perp d_j$ then $d_1 \vee d_2 \vee d_3 \in P$ and $\sigma(d_1 \vee d_2 \vee d_3) = \sigma(d_1) + \sigma(d_2) + \sigma(d_3)$. Let $b = (d_1 \vee d_2 \vee d_3)^\perp$, then $\sigma(b) + \sigma(d_1) + \sigma(d_2) + \sigma(d_3) = 1$. □

Theorem 4.10. *Let P be a poset with the least element* 0 *and the largest element*

1. *A set of maps from P into* $[0, 1]$ *satisfy the following conditions*:
   (i) For $a, b \in P, a \le b$ *iff* $\sigma(a) \le \sigma(b)$, *and* $\sigma(0) = 0, \sigma(1) = 1$, *for any* $\sigma \in \varphi$.
   (ii) For any countable subset $D : d_1, d_2, \cdots, d_n, \cdots$ *of elements of* P, *where* $i \ne j$ *implies* $\sigma(d_i) + \sigma(d_j) \le 1$ *for all* $\sigma \in \varphi$, *there exists* $b \in P$ *such that* $\sigma(b) + \Sigma_{i=1}^{\infty}\sigma(d_i) = 1$ *for all* $\sigma \in \varphi$.

Then $P = (P, \leq, \perp)$, where $a \in P$, $a^{\perp}$ is the unique element of P satisfying $\sigma(a) + \sigma(a^{\perp}) = 1$, for any $\sigma \in \varphi$, is an σ-orthomodular poset and φ is σ-additive order determining system of states on P.

Conversely, if P is a σ-orthomodular poset with a σ-additive order determining system of states φ, then φ satisfies (i), (ii).

Proof: By Theorem 4.9., we know $P = (P, \leq, \perp)$ is an orthomodular poset. We only prove P is σ-complete, that is to say, $\vee_{i=1}^{\infty} a_i \in P$ whenever $a_i \in P$, and $i \neq j, a_i \perp a_j$. Indeed, $i \neq j, a_i \perp a_j$, then $\sigma(a_i) + \sigma(a_j) \leq 1$ for any $\sigma \in \varphi$, by (ii), there exists $b \in P$ such that $\sigma(b^{\perp}) + \Sigma_{i=1}^{\infty} \sigma(a_i) = 1$ for any $\sigma \in P$.

We assert $b = \vee_{i=1}^{\infty} a_i$. For any $i, a_i \leq c, c \in P$, then $a_i \perp c^{\perp}$, by (ii), there exists an element $e \in P$ such that $\sigma(e) + \Sigma_{i=1}^{\infty} \sigma(a_i) + \sigma(c^{\perp}) = 1$ for any $\sigma \in \varphi$, which yields $\Sigma_{i=1}^{\infty} \sigma(a_i) = \sigma(b) = \sigma(c) - \sigma(e) \leq \sigma(c)$, so $b \leq c$, then $b = \vee_{i=1}^{\infty} a_i \in P$. From the above proof, it is easy to see that φ is σ-additive order determining system of states of P.

Conversely, assume φ is an order determining system of states. Clearly, (i) is true. For a countable sequence $d_1, d_2, \cdots, d_i, \cdots$ of elements of P, where $i \neq j, \sigma(d_i) + \sigma(d_j) \leq 1$, for any $\sigma \in \varphi$. Then $\sigma(d_i) \leq \sigma(d_j^{\perp})$, so $d_i \perp d_j$. Define $b^{\perp} = \vee_{i=1}^{\infty} d_i$. Then $\sigma(b^{\perp}) = \Sigma_{i=1}^{\infty} \sigma(d_i)$ implies $\sigma(b) + \Sigma_{i=1}^{\infty} \sigma(d_i) = 1$ for any $\sigma \in \varphi$. □

5. FUZZY EFFECT SPACE

Definition 5.8. (Gudder and Pulmannová, 1997). A binary relation $\sim$ on a partial order abelian monoid P is a weak congruence if it satisfies the following conditions:

(c1) $\sim$ is an equivalence relation,

(c2) $a \perp b, a_1 \perp b_1, a_1 \sim a, b_1 \sim b$ imply $a_1 \oplus b_1 \sim a \oplus b$.
A weak congruence is a congruence if

(c3) $a \perp b, c \sim a$, then there exists $d \in P$ such that $d \sim b, d \perp c$.

Lemma 5.11. (Gudder and Pulmannová, 1997). *Let $\sim$ be a congruence on an effect algebra, and $\sim \neq P \times P$. Then $P/\sim$ is an effect algebra. Where $\tilde{a} = \{b \in P : a \sim b\}$, $\tilde{a} \perp \tilde{b}$ iff there exists $a_1, b_1 \in P$ such that $a_1 \sim a, b_1 \sim b$ and $a_1 \perp b_1$, and we put $\tilde{a} \oplus \tilde{b} = \widetilde{(a_1 \oplus b_1)}$.*

Proposition 5.1. *Let P be an effect algebra with an order determining system of states φ. Define $x \sim y$ iff $\sigma(x) = \sigma(y)$ for all $\sigma \in \varphi$. Then $P\backslash \sim$ is an effect algebra with an order determining system of states.*

Proof: Obviously, $\sim$ is an equivalence relation. Let $a \perp b$, $a_1 \perp b_1$, $a_1 \sim a$, $b_1 \sim b$, then for all $\sigma \in \varphi$, $\sigma(a_1) = \sigma(a)$, $\sigma(b_1) = \sigma(b)$ and $\sigma(a \oplus b) = \sigma(a) + \sigma(b)$, hence $\sigma(a_1 \oplus b_1) = \sigma(a_1) + \sigma(b_1) = \sigma(a \oplus b)$, so $a \oplus b \sim a_1 \oplus b_1$. Let $a \perp b$, $\sigma(c) = \sigma(a)$ for all $\sigma \in \varphi$, then $\sigma(a \oplus b) = \sigma(a) + \sigma(b) = \sigma(c) + \sigma(b) \leq 1$, i.e., $c \leq b^{\perp}$, so $c \oplus b$ is defined in P. Let $d = b$. Then $b \perp c$. Hence, $\sim$ is a congruence on P. Since $(0, 1) \in P \times P$, but $\sigma(0) \neq \sigma(1)$ for all $\sigma \in \varphi$, then 0 is not congruent with 1, i.e., $\sim \subset P \times P$. Thus, by Lemma 5.11., $P/\sim$ is an effect algebra. For any $\sigma \in \varphi$, define $\tilde{\sigma}(\tilde{x}) = \sigma(x)$, for all $\tilde{x} \in P\backslash \sim$. It is easy to see $\{\tilde{\sigma} : \sigma \in \varphi\}$ is an order determining system states on $P\backslash \sim$. $\square$

Remark 5.9. Let $\tilde{P} = P\backslash \sim$, and $\tilde{P} : \tilde{\sigma} \to [0, 1]$ is a state on $\tilde{P}$. Define $\sigma(x) = \tilde{\sigma}(\tilde{x})$, for all $x \in P$, then $\sigma : P \to [0, 1]$ is a state on P. From these, we see that $\tilde{P}$ has close relation with P. Especially, they have the same range for related states, i.e., $\tilde{P}$ preserve the value of state on P.

Lemma 5.12. (Foulis and Bennett, 1994). *Let $(P_\alpha)_{\alpha \in I}$ be a family of effect algebras and let $P = \prod_{\alpha \in P} P_\alpha$ be the cartesian product of the family. Then P can be organized into an effect algebra in such a way that, for $p, q \in P$, $p \oplus q$ is defined in P iff $p_\alpha \oplus q_\alpha$ is defined in P_α for all $\alpha \in I$, in which case $(p \oplus q)_\alpha = p_\alpha \oplus q_\alpha$ for all $\alpha \in I$. This is called the cartesian product of the effect algebra in the family $(P_\alpha)_{\alpha \in I}$. As a special case, if X is a nonempty set and P is an effect algebra, then the set L^X of all functions $\gamma : X \to L$ is again an effect algebra under pointwise operations.*

Example 5.10. Let $L = [0, 1]$, $X \neq \emptyset$, then $[0, 1]^X$ is a lattice effect algebra by Example 2.3. and Lemma 5.12. Moreover, for fuzzy set $f, g \in [0, 1]^X$, $f + g \in [0, 1]^X$ iff $f + g \leq 1_X$ iff $f(x) + g(x) \leq 1$, for all $x \in X$, where 1_X denotes the largest element in $[0, 1]^X$. $f^{\perp} = 1_X - f$, $(f \vee g)(x) = f(x) \vee g(x)$, for all $x \in X$.

Refer to (Foulis and Bennett, 1994), a subset Q of an effect algebra P is called a subeffect algebra of P if (i) $0, 1 \in Q$; (ii) $a \in Q$, then $a^{\perp} \in Q$; (iii) $a, b \in Q$, $a \perp b$, then $a \oplus b \in Q$. Further, if P is a lattice effect algebra and Q is a sublattice of P, then Q is called a sublattice effect algebra of P.

Definition 5.11. A fuzzy topology space $([0, 1]^X, \delta)$ (Liu and Luo, 1998) is called a fuzzy effect space if there exists a base β in δ such that β is a sublattice effect algebra of $[0, 1]^X$. β is called an effect base.

Remark 5.12. Let φ be an order determining system of states of an effect algebra P. If $a \wedge b$ is defined in P, $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$, for all $\sigma \in \varphi$, then φ is called a strong order determining system of states.

Proposition 5.2. *Let P be an effect algebra with an order determining system of states, $a, b \in P$, and $a \neq b$, then there exists a state $\sigma \in \varphi$, such that $\sigma(a) \neq \sigma(b)$.*

Example 5.13. Let $P = [0, 1]^X$. Define $\sigma_{x_0} : P \to [0, 1]$, where $\sigma_{x_0}(A) = A(x_0)$, then P is a lattice effect algebra with a strong order determining system of states $\{\sigma_{x_0} : x_0 \in X\}$.

Definition 5.14. *(Foulis and Bennett, 1994).* Let P,Q be effect algebras. A bijective map μ from P to Q is called an isomorphism if

(i) $\mu(1) = 1$;
(ii) $a, b \in P$, $a \perp b$ iff $\mu(a) \perp \mu(b)$ in which case $\mu(a \oplus b) = \mu(a) \oplus \mu(b)$.

Theorem 5.15. *Let P be a poset. Then P is a lattice effect algebra with a strong order determining system states iff P is isomorphic to an effect base of some fuzzy effect space.*

Proof: Necessary. Let P be a lattice effect algebra, where φ is its strong order determining system of states. Let $\phi = \varphi$, and define $\mu : P \to [0, 1]^\phi$ as follows: $\mu(a)(\sigma) = \sigma(a)$ for all $a \in P, \sigma \in \phi$. Obviously, $\mu(1)(\sigma) \equiv 1$, i.e., $\mu(1)$ is the largest element in $[0, 1]^\phi$. Let $\beta = \{\mu(a) | a \in P\}$, then $(\mu(a) \wedge \mu(b))(\sigma) = \mu(a)(\sigma) \wedge \mu(b)(\sigma) = \sigma(a) \wedge \sigma(b) = \sigma(a \wedge b) = \mu(a \wedge b)(\sigma)$ for all $\sigma \in \phi$. Hence, $\mu(a) \wedge \mu(b) = \mu(a \wedge b)$. That is, β is closed under finite meet. Thus there exists a fuzzy topology δ in ϕ whose base is β.

Let prove β is a sublattice effect algebra of $[0, 1]^\phi$. Since $(\mu(a)^\perp)(\sigma) = 1 - \mu(a)(\sigma) = 1 - \sigma(a) = \sigma(a^\perp) = \mu(a^\perp)(\sigma)$, for all $\sigma \in \phi$, then $\mu(a)^\perp = \mu(a^\perp) \in \beta$. Hence β is a lattice since $\mu(a) \vee \mu(b) = (\mu(a^\perp) \wedge \mu(b^\perp))^\perp$ and β is closed under finite meet and orthocomplement. Let $\mu(a) \perp \mu(b)$, then $\mu(a) + \mu(b) \leq 1_\phi$, that is, $(\mu(a) + \mu(b))(\sigma) = \mu(a)(\sigma) + \mu(b)\sigma = \sigma(a) + \sigma(b) \leq 1$, for all $\sigma \in \phi$. Hence, $a \oplus b$ is defined in P, namely, $(\mu(a) + \mu(b))(\sigma) = \sigma(a) + \sigma(b) = \sigma(a \oplus b) = (\mu(a \oplus b))(\sigma)$. It follows that $\mu(a) + \mu(b) = \mu(a \oplus b) \in \beta$. So β is a sublattice effect algebra of $[0, 1]^\phi$. Thus, β is effect base of δ, and $([0, 1]^\phi, \delta)$ is a fuzzy effect space.

Let prove $\mu : P \to \beta$ is an isomorphism. Obviously, we have proved $\mu(1) = 1_\phi$ and for $a, b \in P$, $\mu(a) \perp \mu(b)$ implies $a \perp b$. Now, if $a \perp b$, then $1 \geq \sigma(a) + \sigma(b) = \mu(a)(\sigma) + \mu(b)(\sigma) = (\mu(a) + \mu(b))(\sigma)$, for all $\sigma \in \phi$, i.e., $\mu(a) + \mu(b) \leq 1_\phi$, hence $\mu(a) \perp \mu(b)$. From the above proof, we see (i), (ii) in Definition 5.14. is satisfied. We only need to prove μ is bijective. Obviously, μ is surjective. For $a, b \in P, a \neq b$, by Proposition 5.2., then there exists $a \nu \in \varphi$ such that $\nu(a) \neq \nu(b)$, hence, $\mu(a)(\nu) = \nu(a) \neq \nu(b) = \mu(b)(\nu)$, that is, μ is monotone. So μ is a isomorphism.

Sufficiency. If P is isomorphic to an effect base β, then P is a lattice effect algebra. Similar to Example 5.13., let $v_x : \beta \to [0, 1]$, where $v_x(B) = B(x)$, for all $x \in X$, $B \in \beta$, then $\{v_x | x \in X\}$ is a strong order determining system of states. Hence, β, i.e., P is a lattice effect algebra with a strong order determining system of states. □

ACKNOWLEDGMENTS

This work is supported by National Science Foundation of China (Grant No:60174016), "TRAPOYT" of China, and the key Project of Fundamental Research (Grant No:2002CB312200).

REFERENCES

Dvurečenskij, A. (1999). Fuzzy set representations of some quantum structures. *Fuzzy Sets and Systems* **101**, 67–78.

Dvurečenskig, A. and Pulmannová, S. (2000). *New Trends in Quantum Structures*, Kluwer, Dordrecht, The Netherlands.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebra and unsharp quantum logics. *International Journal of Theoretical Physics* **24**, 1325–1346.

Giuntini, R. and Greuling, H. (1989). Toward a formal language for unsharp properties. *Foundations of Physics* **19**, 931–945.

Gudder, S. and Pulmannová, S. (1997). Quotients of partial abelian monoids. *Algebra Universalis* **38**, 395–421.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London.

Kôpka, F. (1992). D-Poset of fuzzy sets. *Tatra Mountains Mathematical Publications* **1**, 83–87.

Kôpka, F. and Chovanec, F. (1994). D-Posets. *Mathematical Slovaca* **44**, 21–34.

Liu, Y. M. and Luo, M. K. (1998). *Fuzzy topology*, World Scientific, Singapore.

Mesiar, R. (1994). h-Fuzzy quantum logics. *International Journal of Theoretical Physics* **33**, 1417–1425.

Miklós, R. (1998). *Quantum Logic in Algebraic Approach*, Kluwer Academic, Dodrecht, The Netherlands.

Pykacz, J. (1992). Fuzzy set ideas in quantum logics. *International Journal of Theoretical Physics* **31**, 1767–1783.

Pykacz, J. (1994). Fuzzy quantum logics and infinite-valued lukasiewicz logic. *International Journal of Theoretical Physics* **33**, 1403–1416.